Fuzzy Topology and Łukasiewicz Logics from the Viewpoint of Duality Theory*

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Abstract

This paper explores relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. We first show a fuzzy topological duality for the algebras of Łukasiewicz n-valued logic with truth constants, which generalizes Stone duality for Boolean algebras to the n-valued case via fuzzy topology. Then, based on this duality, we show a fuzzy topological duality for the algebras of modal Łukasiewicz n-valued logic with truth constants, which generalizes Jónsson-Tarski duality for modal algebras to the n-valued case via fuzzy topology. We emphasize that fuzzy topological spaces naturally arise as spectrums of algebras of many-valued logics.

Keywords: fuzzy topology; Stone duality; Jónsson-Tarski duality; algebraic logic; many-valued logic; modal logic; Kripke semantics; compactness

1 Introduction

This paper aims to explore relationships between many-valued logic and fuzzy topology from the viewpoint of duality theory. In particular, we consider fuzzy topological dualities for the algebras of Łukasiewicz n-valued logic $Ł_n^c$ with truth constants and for the algebras of modal Łukasiewicz n-valued logic $MŁ_n^c$ with truth constants.

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Roughly speaking, a many-valued logic is a logical system in which there are more than two truth values (for a general introduction, see [13, 15, 21]). In many-valued logic, a proposition may have a truth value different from 0 (false) and 1 (true). Łukasiewicz many-valued logic is one of the most prominent many-valued logics. Many-valued logics have often been studied from the algebraic point of view (see, e.g., [2, 6, 15]). MV-algebra introduced in [4] provides algebraic semantics for Łukasiewicz infinite-valued logic. MV_n-algebra introduced in [14] provides algebraic semantics for Łukasiewicz n-valued logic introduced in [20] ([14] also gives an axiomatization of Łukasiewicz n-valued logic). Ł_n^c-algebra in this paper is considered as MV_n-algebra enriched by constants.

Kripke semantics for modal logic is naturally extended to the many-valued case by allowing for more than two truth values at each possible world and so we can define modal many-valued logics by such many-valued Kripke semantics, including modal Łukasiewicz many-valued logics. Modal many-valued logics have already been studied by several authors (see [9, 10, 22, 29]).

As a major branch of fuzzy mathematics, fuzzy topology is based on the concept of fuzzy set introduced in [30, 11], which is defined by considering many-valued membership function. For example, a $[0,1]$-valued fuzzy set μ on a set X is defined as a function from X to $[0,1]$. Then, for $x \in X$ and $r \in [0,1]$, $\mu(x) = r$ intuitively means that the proposition "$x \in \mu$" has a truth value r. A fuzzy topology on a set is defined as a collection of fuzzy sets on the set which satisfies some conditions (for details, see Section 3). Historically, Chang [5] introduced the concept of $[0,1]$-valued fuzzy topology and thereafter Goguen [12] introduced that of lattice-valued fuzzy topology. There have been many studies on fuzzy topology (see, e.g., [19, 25, 27]).

Stone duality for Boolean algebras (see [17, 28]) is one of the most important results in algebraic logic and states that there is a categorical duality between Boolean algebras (i.e., the algebras of classical propositional logic) and Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces). Since both many-valued logic and fuzzy topology can be considered as based on the idea that there are more than two truth values, it is natural to expect that there is a duality between the algebras of many-valued logic and "fuzzy Boolean spaces." Stone duality for Boolean algebras was extended to Jónsson-Tarski duality (see [1, 3, 16, 26]) between modal algebras and relational spaces (or descriptive general frames), which is another classical theorem in duality theory. Thus, it is also natural to expect that there is a duality between the algebras of modal many-valued logic and "fuzzy relational spaces."

In this paper, we realize the above expectations in the cases of Ł_n^c and MŁ_n^c. We first develop a categorical duality between the algebras of Ł_n^c and **n**-fuzzy Boolean spaces (see Definition 4.5), which is a generalization of Stone duality for Boolean algebras to the **n**-valued case via fuzzy topology. This duality is developed based on the following insights:

- The spectrum of an algebra of Ł_n^c can be naturally equipped with a certain **n**-fuzzy topology (see Definition 4.9).
- The notion of clopen subset of Boolean space in Stone duality for Boolean algebras corresponds to that of continuous function from **n**-fuzzy Boolean space to **n** ($= \{0, 1/(n-1), 2/(n-1), ..., 1\}$) equipped with the **n**-fuzzy discrete topology in the duality for the algebras of Ł_n^c. This means that the zero-dimensionality of **n**-fuzzy topological spaces is defined in terms of continuous function into **n** (see Definition 4.4).

Moreover, based on the duality for the algebras of Ł_n^c, we develop a categorical duality between the algebras of MŁ_n^c and **n**-fuzzy relational spaces (see Definition 6.3), which is a generalization of Jónsson-Tarski duality for modal algebras to the **n**-valued case via fuzzy topology. Note that an **n**-fuzzy relational space is also defined in terms of continuous functions into **n** (see the items 1 and 2 in the object part of Definition 6.3).

There have been some studies on dualities for algebras of many-valued logics (see, e.g., [2, 7, 18, 23, 24, 8, 29]). However, they are based on the ordinary topology and therefore do not reveal relationships between many-valued logic and fuzzy topology. By the results in this paper, we can notice that fuzzy topological spaces naturally arise as spectrums of algebras of some many-valued logics and that there are categorical dualities connecting fuzzy topology and those many-valued logics which generalize Stone and Jónsson-Tarski dualities via fuzzy topology.

This paper is organized as follows. In Section 2, we define Ł_n^c and Ł_n^c-algebras, and show basic properties of them. In Section 3, we review basic concepts related to fuzzy topology. In Section 4, we define **n**-fuzzy Boolean spaces and show a fuzzy topological duality for Ł_n^c-algebras, which is a main theorem in this paper. In Section 5, we define MŁ_n^c and MŁ_n^c-algebras, and show basic properties of them, including a compactness theorem for MŁ_n^c. In Section 6, we define **n**-fuzzy relational spaces and show a fuzzy topological duality for MŁ_n^c-algebras, which is the other main theorem.

2 $Ł_n^c$-algebras and basic properties

Throughout this paper, n denotes a natural number more than 1.

Definition 2.1. $\mathbf{n}$ denotes $\{0, 1/(n-1), 2/(n-1), ..., 1\}$. We equip $\mathbf{n}$ with all constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp})$ defined as follows:

$$
\begin{aligned}
x \wedge y &= \min(x, y);\\
x \vee y &= \max(x, y);\\
x * y &= \max(0, x + y - 1);\\
x \wp y &= \min(1, x + y);\\
x \rightarrow y &= \min(1, 1 - (x - y));\\
x^{\perp} &= 1 - x.
\end{aligned}
$$

We define Łukasiewicz n-valued logic with truth constants, which is denoted by $Ł_n^c$. The connectives of $Ł_n^c$ are

$$(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1),$$

where $(\wedge, \vee, *, \wp, \rightarrow)$ are binary connectives, $(\text{-})^{\perp}$ is a unary connective, and $(0, 1/(n-1), 2/(n-1), ..., 1)$ are constants. The formulas of $Ł_n^c$ are recursively defined in the usual way. Let **PV** denote the set of propositional variables and **Form** denote the set of formulas of $Ł_n^c$.

$x \leftrightarrow y$ is the abbreviation of $(x \rightarrow y) \wedge (y \rightarrow x)$. For $m \in \omega$ with $m \neq 0$, $*^m x$ is the abbreviation of $x * ... * x$ (m-times). For instance, $*^3 x = x * x * x$.

Definition 2.2. A function $v : \mathbf{Form} \rightarrow \mathbf{n}$ is an $\mathbf{n}$-valuation iff it satisfies:

- $v(\varphi @ \psi) = v(\varphi) @ v(\psi)$ for $@ = \wedge, \vee, *, \wp, \rightarrow$;
- $v(\varphi^{\perp}) = (v(\varphi))^{\perp}$;
- $v(r) = r$ for $r \in \mathbf{n}$.

Define $Ł_n^c = \{\varphi \in \mathbf{Form}\ ;\ v(\varphi) = 1 \text{ for any } \mathbf{n}\text{-valuation } v\}$.

$Ł_n^c$-algebras and homomorphisms are defined as follows.

Definition 2.3. $(A, \wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$ is an $Ł_n^c$-algebra iff it satisfies the following set of equations: $\{\varphi = \psi\ ;\ \varphi \leftrightarrow \psi \in Ł_n^c\}$.

A homomorphism of $Ł_n^c$-algebras is defined as a function which preserves the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$.

Throughout this paper, we do not distinguish between formulas of $Ł_n^c$ and terms of $Ł_n^c$-algebras.

Definition 2.4. $\varphi \in \mathbf{Form}$ is idempotent iff $\varphi * \varphi \leftrightarrow \varphi \in Ł_n^c$.

For an $Ł_n^c$-algebra A, $a \in A$ is idempotent iff $a * a = a$.

$\mathcal{B}(A)$ denotes the set of all idempotent elements of an $Ł_n^c$-algebra A.

Let A be an $Ł_n^c$-algebra. Then, we have the following facts: (i) For $a \in A$, $*^{n-1}a$ is always idempotent. (ii) If $a \in A$ is idempotent, then either $v(a) = 1$ or $v(a) = 0$ holds for any homomorphism $v : A \to \mathbf{n}$. (iii) If $a, b \in A$ are idempotent, then $a * b = (*^{n-1}a) * (*^{n-1}b) = (*^{n-1}a) \wedge (*^{n-1}b) = a \wedge b$ and $a \wp b = (*^{n-1}a) \wp (*^{n-1}b) = (*^{n-1}a) \vee (*^{n-1}b) = a \vee b$.

It is easy to verify the following:

Proposition 2.5. *For an $Ł_n^c$-algebra A, $\mathcal{B}(A)$ forms a Boolean algebra. In particular, $a \vee a^{\perp} = 1$ for any idempotent element a of A.*

In the following, we define a formula $\mathrm{T}_r(x)$ for $r \in \mathbf{n}$, which intuitively means that the truth value of x is exactly r.

Lemma 2.6. *Let A be an $Ł_n^c$-algebra and $r \in \mathbf{n}$. There is an idempotent formula $\mathrm{T}_r(x)$ with one variable x such that, for any homomorphism $v : A \to \mathbf{n}$ and any $a \in A$, the following hold:*

- $v(\mathrm{T}_r(a)) = 1$ *iff* $v(a) = r$;
- $v(\mathrm{T}_r(a)) = 0$ *iff* $v(a) \neq r$.

Proof. If $r = 0$, then we can set $\mathrm{T}_r(x) = *^{n-1}(x^{\perp})$. If $r = 1$, then we can set $\mathrm{T}_r(x) = *^{n-1}x$.

Let $r = k/(n-1)$ for $k \in \{1, ..., n-2\}$. If k is a divisor of $n-1$, then we can set

$$\mathrm{T}_r(x) = *^{n-1}(x \leftrightarrow (\wp^{\frac{n-1}{k}-1}x)^{\perp}).$$

For a rational number q, let $[q]$ denote the greatest integer n such that $n \leq q$. If k is not a divisor of $n-1$, then

$$\begin{aligned} v(x) = k/(n-1) \quad &\text{iff} \quad v(\wp^{[\frac{n-1}{k}]}x) = \frac{k}{n-1}\left[\frac{n-1}{k}\right] (< 1) \\ &\text{iff} \quad v((\wp^{[\frac{n-1}{k}]}x)^{\perp}) = 1 - \frac{k}{n-1}\left[\frac{n-1}{k}\right]. \end{aligned}$$

Since

$$1 - \frac{k}{n-1}\left[\frac{n-1}{k}\right] < \frac{k}{n-1},$$

this lemma follows by induction on k. □

The above lemma is more easily proved by using truth constants $r \in \mathbf{n}$. However, it must be stressed that the above proof works even if we consider Łukasiewicz n-valued logic without truth constants.

Note that any homomorphism preserves the operation T_r(-).

Lemma 2.7. *Let A be an $Ł_n^c$-algebra and $a_i \in A$ for a finite set I and $i \in I$. Then, (i) $\mathrm{T}_1(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \mathrm{T}_1(a_i)$; (ii) $\mathrm{T}_1(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} \mathrm{T}_1(a_i)$.*

Proof. Since $\mathbf{n}$ is totally ordered, we have (i). (ii) is immediate. □

By (ii) in the above lemma, T_1(-) is order preserving.

Lemma 2.8. *Let A be an $Ł_n^c$-algebra and $r \in \mathbf{n}$. There is an idempotent formula $\mathrm{U}_r(x)$ with one variable x such that, for any homomorphism $v : A \to \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(\mathrm{U}_r(a)) = 1$ iff $v(a) \geq r$; (ii) $v(\mathrm{U}_r(a)) = 0$ iff $v(a) \ngeq r$.*

Proof. It suffices to let $\mathrm{U}_r(x) = \bigvee\{\mathrm{T}_s(x)\,;\, r \leq s\}$ by Lemma 2.6. □

Note that any homomorphism preserves the operation U_r(-).

Lemma 2.9. *Let A be an $Ł_n^c$-algebra and $r \in \mathbf{n}$. There is a formula $S_r(x)$ with one variable x such that, for any homomorphism $v : A \to \mathbf{n}$ and any $a \in A$, the following two conditions hold: (i) $v(S_r(a)) = r$ iff $v(a) = 1$; (ii) $v(S_r(a)) = 0$ iff $v(a) \neq 1$.*

Proof. Let $S_r(x) = (\mathrm{T}_1(x) \to r) \wedge ((\mathrm{T}_1(x))^{\perp} \to 0)$. □

Note that any homomorphism preserves the operation S_r(-).

Lemma 2.10. *Let A be an $Ł_n^c$-algebra. Let v and u be homomorphisms from A to $\mathbf{n}$. Then, (i) $v = u$ iff (ii) $v^{-1}(\{1\}) = u^{-1}(\{1\})$.*

Proof. Clearly, (i) implies (ii). We show the converse. Assume that $v^{-1}(\{1\}) = u^{-1}(\{1\})$. Suppose for contradiction that $v(a) \neq u(a)$ for some $a \in A$. Let $r = v(a)$. Then $v(\mathrm{T}_r(a)) = 1$ and $u(\mathrm{T}_r(a)) = 0$, which contradicts $v^{-1}(\{1\}) = u^{-1}(\{1\})$. □

For an $Ł_n^c$-algebra A and $a, b \in A$, we mean $a \vee b = b$ by $a \leq b$.

Lemma 2.11. *Let A be an $Ł_n^c$-algebra. For any $a, b \in A$, the following holds:*

$$\bigwedge_{r\in\mathbf{n}} (\mathrm{T}_r(a) \leftrightarrow \mathrm{T}_r(b)) \leq a \leftrightarrow b.$$

Proof. This is proved by straightforward computation. $\square$

For a partially ordered set $(M, \leq)$, $X \subset M$ is called an upper set iff if $x \in X$ and $x \leq y$ for $y \in M$ then $y \in X$.

Definition 2.12. Let A be an Ł_n^c-algebra. A non-empty subset F of A is called an **n**-filter of A iff F is an upper set and is closed under $*$. An **n**-filter F of A is called proper iff $F \neq A$.

An **n**-filter of A is closed under $\wedge$, since $a * b \leq a \wedge b$ for any $a, b \in A$.

Definition 2.13. Let A be an Ł_n^c-algebra. A proper **n**-filter P of A is prime iff, for any $a, b \in A$, $a \vee b \in P$ implies either $a \in P$ or $b \in P$.

Proposition 2.14. *Let A be an Ł_n^c-algebra and F an **n**-filter of A. For $b \in A$, assume $b \notin F$. Then, there is a prime **n**-filter P of A such that $F \subset P$ and $b \notin P$.*

Proof. Let Z be the set of all those **n**-filters G of A such that $F \subset G$ and $b \notin G$. Then $F \in Z$. Clearly, every chain of Z has an upper bound in Z. Thus, by Zorn's lemma, we have a maximal element P in Z. Note that $F \subset P$ and $b \notin P$.

To complete the proof, it suffices to show that P is a prime **n**-filter of A. Assume $x \vee y \in P$. Additionally, suppose for contradiction that $x \notin P$ and $y \notin P$. Then, since P is maximal, there exists $\varphi_x \in A$ such that $\varphi_x \leq b$ and $\varphi_x = (*^{n-1}x) * p_x$ for some $p_x \in P$. Similary, there exists $\varphi_y \in A$ such that $\varphi_y \leq b$ and $\varphi_y = (*^{n-1}y) * p_y$ for some $p_y \in P$. Now, we have the following:

$$\begin{aligned}
b &\geq ((*^{n-1}x) * p_x) \vee ((*^{n-1}y) * p_y) \\
&\geq (*^{n-1}(x * p_x)) \vee (*^{n-1}(y * p_y)) \\
&= *^{n-1}((x * p_x) \vee (y * p_y)) \\
&\geq *^{n-1}((x \vee (y * p_y)) * (p_x \vee (y * p_y))) \\
&\geq *^{n-1}((x \vee y) * p_y * p_x),
\end{aligned}$$

where note that $*^{n-1}(x \vee y) = (*^{n-1}x) \vee (*^{n-1}y)$ and $x \vee (y * z) \geq (x \vee y) * (x \vee z)$ for any $x, y, z \in A$. Since $p_x, p_y, x \vee y \in P$, we have $b \in P$, which is a contradiction. Hence P is a prime **n**-filter of A. $\square$

We do not use $(\text{-})^\perp$ or $\rightarrow$ in the above proof and therefore the above proof works even for algebras of "intuitionistic Łukasiewicz n-valued logic."

Definition 2.15. Let A be an Ł_n^c-algebra. A subset X of A has finite intersection property (f.i.p.) with respect to $*$ iff, for any $n \in \omega$ with $n \neq 0$, if $a_1, ..., a_n \in X$ then $a_1 * ... * a_n \neq 0$.

Corollary 2.16. *Let A be an $Ł_n^c$-algebra and X a subset of A. If X has f.i.p. with respect to $*$, then there is a prime* **n**-*filter P of A with $X \subset P$.*

Proof. By the assumption, we have a proper **n**-filter F of A generated by X. By letting $b = 0$ in Proposition 2.14, we have a prime **n**-filter P of A with $X \subset P$. □

Proposition 2.17. *Let A be an $Ł_n^c$-algebra. For a prime* **n**-*filter P of A, define $v_P : A \to \mathbf{n}$ by $v_P(a) = r \Leftrightarrow \mathrm{T}_r(a) \in P$. Then, v_P is a bijection from the set of all prime* **n**-*filters of A to the set of all homomorphisms from A to* **n** *with $v_P^{-1}(\{1\}) = P$.*

Proof. Note that v_P is well-defined as a function. We prove that v_P is a homomorphism. We first show $v_P(a * b) = v_P(a) * v_P(b)$ for $a, b \in A$. Let $r = v_P(a)$ and $s = v_P(b)$. Then $\mathrm{T}_r(a) \in P$ and $\mathrm{T}_s(b) \in P$. It is easy to see that $\mathrm{T}_r(a) \wedge \mathrm{T}_s(b) \leq \mathrm{T}_{r*s}(a * b)$, which intuitively means that if the truth value of a is r and if the truth value of b is s then the truth value of $a * b$ is $r * s$. Since $\mathrm{T}_r(a) \in P$ and $\mathrm{T}_s(b) \in P$, we have $\mathrm{T}_{r*s}(a * b) \in P$, whence we have $v_P(a * b) = r * s = v_P(a) * v_P(b)$.

Next we show that $v_P(a^\perp) = v_P(a)^\perp$. Let $r = v_P(a)$. It is easy to see that $\mathrm{T}_r(a) \leq \mathrm{T}_{r^\perp}(a^\perp)$. By $\mathrm{T}_r(a) \in P$, we have $\mathrm{T}_{r^\perp}(a^\perp) \in P$, whence $v_P(a^\perp) = r^\perp = v_P(a)^\perp$. As is well-known, $(\wedge, \vee, \wp, \to)$ can be defined by using only $(*, (\text{-})^\perp)$ (see [6]) and so v_P preserves the operations $(\wedge, \vee, \wp, \to)$. Clearly, v_P preserves any constant $r \in \mathbf{n}$. Thus, v_P is a homomorphism. The remaining part of the proof is straightforward. □

3 n-valued fuzzy topology

Let us review basic concepts from fuzzy set theory and fuzzy topology.

3.1 n-valued fuzzy set theory

An **n**-fuzzy set on a set S is defined as a function from S to **n**. For **n**-fuzzy sets μ, λ on S, define an **n**-fuzzy set $\mu @ \lambda$ on S by $(\mu @ \lambda)(x) = \mu(x) @ \lambda(y)$ for $@ = \wedge, \vee, *, \wp, \to$, and define an **n**-fuzzy set $\mu^\perp$ on S by $(\mu^\perp)(x) = (\mu(x))^\perp$.

Let X, Y be sets and f a function from X to Y. For an **n**-fuzzy set μ on X, define the direct image $f(\mu) : Y \to \mathbf{n}$ of μ under f by

$$f(\mu)(y) = \bigvee \{\mu(x)\,;\, x \in f^{-1}(\{y\})\} \text{ for } y \in Y.$$

For $f : X \to Y$ and an $\mathbf{n}$-fuzzy set λ on Y, define the inverse image $f^{-1}(\lambda) : X \to \mathbf{n}$ of λ under f by $f^{-1}(\lambda) = \lambda \circ f$. Note that f^{-1} commutes with $\bigvee$, i.e., $f^{-1}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} f^{-1}(\mu_i)$ for $\mathbf{n}$-fuzzy sets μ_i on Y.

For a relation R on a set S and an $\mathbf{n}$-fuzzy set μ on S, define an $\mathbf{n}$-fuzzy set $R^{-1}[\mu]$ on S, which is called the inverse image of μ under R, by $R^{-1}[\mu](x) = \bigvee\{\mu(y) \ ; \ xRy\}$ for $x \in S$. Note that $R^{-1}[\bigvee_{i \in I} \mu_i] = \bigvee_{i \in I}(R^{-1}[\mu_i])$.

3.2 n-valued fuzzy topology

For sets X and Y, Y^X denotes the set of all functions from X to Y. We do not distinguish between $r \in \mathbf{n}$ and the constant function whose value is always r.

Definition 3.1 ([30, 12, 27])**.** For a set S and a subset $\mathcal{O}$ of $\mathbf{n}^S$, $(S, \mathcal{O})$ is an $\mathbf{n}$-fuzzy space iff the following hold:

- $r \in \mathcal{O}$ for any $r \in \mathbf{n}$;
- if $\mu_1, \mu_2 \in \mathcal{O}$ then $\mu_1 \wedge \mu_2 \in \mathcal{O}$;
- if $\mu_i \in \mathcal{O}$ for $i \in I$ then $\bigvee_{i \in I} \mu_i \in \mathcal{O}$,

Then, we call $\mathcal{O}$ the $\mathbf{n}$-fuzzy topology of $(S, \mathcal{O})$, and an element of $\mathcal{O}$ an open $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$. An $\mathbf{n}$-fuzzy set λ on S is a closed $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$ iff $\lambda = \mu^{\perp}$ for some open $\mathbf{n}$-fuzzy set μ on $(S, \mathcal{O})$. A clopen $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$ means a closed and open $\mathbf{n}$-fuzzy set on $(S, \mathcal{O})$.

An $\mathbf{n}$-fuzzy space $(S, \mathcal{O})$ is often denoted by its underlying set S.

Definition 3.2. For a set S, $\mathbf{n}^S$ is called the discrete $\mathbf{n}$-fuzzy topology on S. $(S, \mathbf{n}^S)$ is called a discrete $\mathbf{n}$-fuzzy space.

Definition 3.3. Let S_1 and S_2 be $\mathbf{n}$-fuzzy spaces. Then, $f : S_1 \to S_2$ is continuous iff, for any open $\mathbf{n}$-fuzzy set μ on S_2, $f^{-1}(\mu)$ (i.e., $\mu \circ f$) is an open $\mathbf{n}$-fuzzy set on S_1.

A composition of continuous functions between $\mathbf{n}$-fuzzy spaces is also continuous (as a function between $\mathbf{n}$-fuzzy spaces).

Definition 3.4. Let $(S, \mathcal{O})$ be an $\mathbf{n}$-fuzzy space. Then, an open basis $\mathcal{B}$ of $(S, \mathcal{O})$ is a subset of $\mathcal{O}$ such that the following holds: (i) $\mathcal{B}$ is closed under $\wedge$; (ii) for any $\mu \in \mathcal{O}$, there are $\mu_i \in \mathcal{B}$ for $i \in I$ with $\mu = \bigvee_{i \in I} \mu_i$.

Definition 3.5. An $\mathbf{n}$-fuzzy space S is Kolmogorov iff, for any $x, y \in S$ with $x \neq y$, there is an open $\mathbf{n}$-fuzzy set μ on S with $\mu(x) \neq \mu(y)$.

Definition 3.6. An $\mathbf{n}$-fuzzy space S is Hausdorff iff, for any $x, y \in S$ with $x \neq y$, there are $r \in \mathbf{n}$ and open $\mathbf{n}$-fuzzy sets μ, λ on S such that $\mu(x) \geq r$, $\lambda(y) \geq r$ and $\mu \wedge \lambda < r$.

Definition 3.7 ([12])**.** Let S be an $\mathbf{n}$-fuzzy space. An $\mathbf{n}$-fuzzy set λ on S is compact iff, if $\lambda \leq \bigvee_{i \in I} \mu_i$ for open $\mathbf{n}$-fuzzy sets μ_i on S, then there is a finite subset J of I such that $\lambda \leq \bigvee_{i \in J} \mu_i$.

Let 1 denote the constant function on S whose value is always 1. Then, S is compact iff, if $1 = \bigvee_{i \in I} \mu_i$ for open $\mathbf{n}$-fuzzy sets μ_i on S, then there is a finite subset J of I such that $1 = \bigvee_{i \in J} \mu_i$.

We can construct an operation $(\text{-})^*$ which turns an $\mathbf{n}$-fuzzy space into a topological space (in the classical sense) as follows.

Definition 3.8. Let $(S, \mathcal{O})$ be an $\mathbf{n}$-fuzzy space. Define

$$\mathcal{O}^* = \{\mu^{-1}(\{1\})\ ;\ \mu \in \mathcal{O}\}.$$

Then, S^* denotes a topological space $(S, \mathcal{O}^*)$ (see the below proposition).

Lemma 3.9. *Let $(S, \mathcal{O})$ be an $\mathbf{n}$-fuzzy space. Then, S^* forms a topological space.*

Proof. Since $0 \in \mathcal{O}$ and $\emptyset = 0^{-1}(\{1\})$, we have $\emptyset \in \mathcal{O}^*$. Similarly, $S \in \mathcal{O}^*$. Assume $X_i \in \mathcal{O}$ for $i \in I$. Then, $X_i = {\mu_i}^{-1}(\{1\})$ for some $\mu_i \in \mathcal{O}$. Since $\mathbf{n}$ is totally ordered, $\bigcup_{i \in I} X_i = (\bigvee_{i \in I} \mu_i)^{-1}(\{1\})$. Thus, by $\bigvee_{i \in I} \mu_i \in \mathcal{O}$, we have $\bigcup_{i \in I} X_i \in \mathcal{O}^*$. It is easy to verify that $X, Y \in \mathcal{O}$ implies $X \cap Y \in \mathcal{O}^*$. $\Box$

4 A fuzzy topological duality for $Ł_n^c$-algebras

In this section, we show a fuzzy topological duality for $Ł_n^c$-algebras, which is a generalization of Stone duality for Boolean algebras via fuzzy topology, where note that $Ł_2^c$-algebras coincide with Boolean algebras.

Definition 4.1. $Ł_n^c$-Alg denotes the category whose objects are $Ł_n^c$-algebras and whose arrows are homomorphisms of $Ł_n^c$-algebras.

Our aim in this section is to show that the category $Ł_n^c$-Alg is dually equivalent to the category FBS_n, which is defined in the following subsection.

4.1 Category FBS_n

We equip $\mathbf{n}$ with the discrete $\mathbf{n}$-fuzzy topology.

Definition 4.2. Let S be an $\mathbf{n}$-fuzzy space. Then, $\mathrm{Cont}(S)$ is defined as the set of all continuous functions from S to $\mathbf{n}$. We endow $\mathrm{Cont}(S)$ with the operations $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), 2/(n-1), ..., 1)$ defined pointwise: For $f, g \in \mathrm{Cont}(S)$, define $(f@g)(x) = f(x)@g(x)$, where $@ = \wedge, \vee, *, \wp, \rightarrow$. For $f \in \mathrm{Cont}(S)$, define $f^{\perp}(x) = (f(x))^{\perp}$. Finally, $r \in \mathbf{n}$ is defined as the constant function on S whose value is always r.

We show that the operations of $\mathrm{Cont}(S)$ are well-defined:

Lemma 4.3. *Let S be an $\mathbf{n}$-fuzzy space. Then, $\mathrm{Cont}(S)$ is closed under the operations* $(\wedge, \vee, *, \wp, \rightarrow, (\text{-})^{\perp}, 0, 1/(n-1), ..., (n-2)/(n-1), 1)$

Proof. For any $r \in \mathbf{n}$, a constant function $r : S \to \mathbf{n}$ is continuous, since any $s \in \mathbf{n}$ is an open $\mathbf{n}$-fuzzy set on S by Definition 3.1. Then it suffices to show that, if $f, g \in \mathrm{Cont}(S)$, then $f^{\perp}$ and $f@g$ are continuous for $@ = \wedge, \vee, *, \wp, \rightarrow$. Throughout this proof, let $f, g \in \mathrm{Cont}(S)$ and μ an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$, i.e., a function from $\mathbf{n}$ to $\mathbf{n}$. For $r \in \mathbf{n}$, define $\mu_r : \mathbf{n} \to \mathbf{n}$ by

$$\mu_r(x) = \begin{cases} \mu(r) & \text{if } x = r \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\mu = \bigvee_{r \in \mathbf{n}} \mu_r$.

We show that $(f^{\perp})^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set on S. Now, we have

$$(f^{\perp})^{-1}(\mu) = (f^{\perp})^{-1}(\bigvee_{r \in \mathbf{n}} \mu_r) = \bigvee_{r \in \mathbf{n}} ((f^{\perp})^{-1}(\mu_r)).$$

Thus it suffices to show that $(f^{\perp})^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. Define $\lambda_r : \mathbf{n} \to \mathbf{n}$ by

$$\lambda_r(x) = \begin{cases} \mu(r) & \text{if } x = 1 - r \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that $(f^{\perp})^{-1}(\mu_r) = f^{-1}(\lambda_r)$. Since f is continuous and since λ_r is an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$, $f^{-1}(\lambda_r)$ is an open $\mathbf{n}$-fuzzy set on S.

Next, we show that $(f * g)^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set on S. By the same argument as in the case of $f^{\perp}$, it suffices to show that $(f * g)^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. For $p \in \mathbf{n}$, define $\theta_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\theta_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = p \\ 0 & \text{otherwise.} \end{cases}$$

For $r \neq 0$, define $\kappa_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x = r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 0$, define $\kappa_{r,p} : \mathbf{n} \to \mathbf{n}$ by

$$\kappa_{r,p}(x) = \begin{cases} \mu(r) & \text{if } x \leq r - p + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that

$$(f * g)^{-1}(\mu_r) = \bigvee_{p \in \mathbf{n}} (f^{-1}(\theta_{r,p}) \wedge g^{-1}(\kappa_{r,p})).$$

Since $f, g \in \mathrm{Cont}(S)$, the right-hand side is an open $\mathbf{n}$-fuzzy set on S.

As is well-known, $(\wedge, \vee, \wp, \to)$ can be defined by using only $(*, (\text{-})^{\perp})$ (see [6]) and so $(f@g)^{-1}(\mu)$ is an open $\mathbf{n}$-fuzzy set for $@ = \wedge, \vee, \wp, \to$. $\square$

Definition 4.4. For an $\mathbf{n}$-fuzzy space S, S is zero-dimensional iff $\mathrm{Cont}(S)$ forms an open basis of S.

Definition 4.5. For an $\mathbf{n}$-fuzzy space S, S is an $\mathbf{n}$-fuzzy Boolean space iff S is zero-dimensional, compact and Kolmogorov.

Definition 4.6. FBS_n is defined as the category of $\mathbf{n}$-fuzzy Boolean spaces and continuous functions.

Proposition 4.7. *Let S be an $\mathbf{n}$-fuzzy space. Then, (i) S is an $\mathbf{n}$-fuzzy Boolean space iff (ii) S is zero-dimensional, compact and Hausdorff.*

Proof. Cleary, (ii) implies (i). We show the converse. Assume that S is an $\mathbf{n}$-fuzzy Boolean space. It suffices to show that S is Hausdorff. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and since S is zero-dimensional, there is $\mu \in \mathrm{Cont}(S)$ with $\mu(x) \neq \mu(y)$. Let $s = \mu(x)$. Then, $\mathrm{T}_s \circ \mu(x) = 1$ and $(\mathrm{T}_s \circ \mu)^{\perp}(y) = 1$. Since $\mathrm{T}_s : \mathbf{n} \to \mathbf{n}$ is continuous, $\mathrm{T}_s \circ \mu \in \mathrm{Cont}(S)$ and $(\mathrm{T}_s \circ \mu)^{\perp} \in \mathrm{Cont}(S)$ by Lemma 4.3. Since S is zero-dimensional, $\mathrm{T}_s \circ \mu$ and $(\mathrm{T}_s \circ \mu)^{\perp}$ are open $\mathbf{n}$-fuzzy sets on S. We also have $(\mathrm{T}_s \circ \mu) \wedge (\mathrm{T}_s \circ \mu)^{\perp} = 0$. Thus, S is Hausdorff. $\square$

Next we show that (-)* turns an $\mathbf{n}$-fuzzy Boolean space into a Boolean space, i.e., a zero-dimensional compact Hausdorff space.

Proposition 4.8. *Let S be an $\mathbf{n}$-fuzzy Boolean space. Then, S^* forms a Boolean space.*

Proof. By Lemma 3.9, S^* is a topological space.

First, we show that S^* is zero-dimensional in the classical sense. Let $\mathcal{B}^* = \{\mu^{-1}(\{1\})\ ;\ \mu \in \mathrm{Cont}(S)\}$, where, since S is zero-dimensional and so $\mu \in \mathrm{Cont}(S)$ is an open $\mathbf{n}$-fuzzy set on S, $\mu^{-1}(\{1\})$ is an open subset of S^*. We claim that $\mathcal{B}^*$ forms an open basis of S^*. It is easily verified that $\mathcal{B}^*$ is closed under $\cap$. Assume that O is an open subset of S^*, i.e., $O = \mu^{-1}(\{1\})$ for some open $\mathbf{n}$-fuzzy set μ on S. Since S is zero-dimensional, there are $\mu_i \in \mathrm{Cont}(S)$ with $\mu = \bigvee_{i\in I}\mu_i$. Since $\mathbf{n}$ is totally ordered, $O = \bigcup_{i\in I}\mu_i^{-1}(\{1\})$. It follows from $\mu_i \in \mathrm{Cont}(S)$ that $\mu_i^{-1}(\{1\}) \in \mathcal{B}^*$ for any $i \in I$. This completes the proof of the claim. If $\mu \in \mathrm{Cont}(S)$, then

$$(\mu^{-1}(\{1\}))^c = ((\mathrm{T}_1 \circ \mu)^{\perp})^{-1}(\{1\}).$$

Since $\mathrm{T}_1 : \mathbf{n} \to \mathbf{n}$ is continuous, $\mathrm{T}_1 \circ \mu \in \mathrm{Cont}(S)$, whence, by Lemma 4.3, $(\mathrm{T}_1 \circ \mu)^{\perp} \in \mathrm{Cont}(S)$. Thus the right-hand side is open in S^* and so $\mu^{-1}(\{1\})$ is clopen in S^* for $\mu \in \mathrm{Cont}(S)$. Hence, S^* is zero-dimensional.

Second, we show that S^* is compact in the classical sense. Assume that $S^* = \bigcup_{i\in I} O_i$ for some open subsets O_i of S^*. Since $\mathcal{B}^*$ forms an open basis of S^*, we may assume that $S^* = \bigcup_{i\in I}\mu_i^{-1}(\{1\})$ for some $\mu_i \in \mathrm{Cont}(S)$. Then, $1 = \bigvee_{i\in I}\mu_i$ where 1 denotes the constant function on S $(= S^*)$ whose value is always 1. Since S is zero-dimensional, μ_i is an open $\mathbf{n}$-fuzzy set on S. Thus, since S is compact, there is a finite subset J of I such that $1 = \bigvee_{j\in J}\mu_j$, whence $S^* = \bigcup_{j\in J}\mu_j^{-1}(\{1\})$. Hence S^* is compact.

Finally, we show that S^* is Hausdorff in the classical sense. Since S^* is zero-dimensional, it suffices to show that S^* is Kolmogorov in the classical sense. Assume $x, y \in S^*$ with $x \neq y$. Since S is Kolmogorov, there is an open $\mathbf{n}$-fuzzy set μ on S with $\mu(x) \neq \mu(y)$. Since S is zero-dimensional, $\mu = \bigvee_{i\in I}\mu_i$ for some $\mu_i \in \mathrm{Cont}(S)$. There is $i \in I$ with $\mu_i(x) \neq \mu_i(y)$. Let $r = \mu_i(x)$. Then, we have $\mathrm{T}_r \circ \mu_i(x) = 1$ and $\mathrm{T}_r \circ \mu_i(y) = 0$, whence we have $x \in (\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$ and $y \notin (\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$. Since $\mathrm{T}_r : \mathbf{n} \to \mathbf{n}$ is continuous, it follows from $\mu_i \in \mathrm{Cont}(S)$ that $\mathrm{T}_r \circ \mu_i \in \mathrm{Cont}(S)$, whence $\mathrm{T}_r \circ \mu_i$ is an open $\mathbf{n}$-fuzzy set on S and so $(\mathrm{T}_r \circ \mu_i)^{-1}(\{1\})$ is an open subset of S^*. Hence S^* is Kolmogorov. $\square$

4.2 Functors Spec and Cont

We define the spectrum $\mathrm{Spec}(A)$ of an Ł_n^c-algebra A as follows.

Definition 4.9. For an $Ł_n^c$-algebra A, $\mathrm{Spec}(A)$ is defined as the set of all homomorphisms (of $Ł_n^c$-algebras) from A to $\mathbf{n}$ equipped with the $\mathbf{n}$-fuzzy topology generated by $\{\langle a\rangle\ ;\ a \in A\}$, where $\langle a\rangle : \mathrm{Spec}(A) \to \mathbf{n}$ is defined by

$$\langle a\rangle(v) = v(a).$$

The operations $(\wedge, \vee, *, \wp, \to, (\text{-})^{\perp})$ on $\{\langle a\rangle\ ;\ a \in A\}$ are defined pointwise as in Definition 4.2.

$\{\langle a\rangle\ ;\ a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$, since $\langle a\rangle \wedge \langle b\rangle = \langle a \wedge b\rangle$.

Definition 4.10. We define a contravariant functor $\mathrm{Spec} : Ł_n^c\text{-Alg} \to \mathrm{FBS}_n$.

For an object A in $Ł_n^c$-Alg, define $\mathrm{Spec}(A)$ as in Definition 4.9.

For an arrow $f : A_1 \to A_2$ in $Ł_n^c$-Alg, define $\mathrm{Spec}(f) : \mathrm{Spec}(A_2) \to \mathrm{Spec}(A_1)$ by $\mathrm{Spec}(f)(v) = v \circ f$ for $v \in \mathrm{Spec}(A_2)$.

The well-definedness of the functor Spec is proved by Proposition 4.15 and Proposition 4.16 below.

Since $\mathbf{n}$ is a totally ordered complete lattice, we have:

Lemma 4.11. *Let μ_i be an $\mathbf{n}$-fuzzy set on a set S for a set I and $i \in I$. Then, (i) $\mathrm{T}_1 \circ \bigvee_{i\in I} \mu_i = \bigvee_{i\in I}(\mathrm{T}_1 \circ \mu_i)$; (ii) $\mathrm{T}_1 \circ \bigwedge_{i\in I} \mu_i = \bigwedge_{i\in I}(\mathrm{T}_1 \circ \mu_i)$.*

Lemma 4.12. *Let A be an $Ł_n^c$-algebra. Then, $\mathrm{Spec}(A)$ is compact.*

Proof. Assume that $1 = \bigvee_{j\in J} \mu_j$ for open $\mathbf{n}$-fuzzy sets μ_j on $\mathrm{Spec}(A)$, where 1 denotes the constant function defined on $\mathrm{Spec}(A)$ whose value is always 1. Then, since $\{\langle a\rangle\ ;\ a \in A\}$ is an open basis of $\mathrm{Spec}(A)$, we may assume that $1 = \bigvee_{i\in I}\langle a_i\rangle$ for some $a_i \in A$. It follows from Lemma 4.11 that $1 = \mathrm{T}_1 \circ 1 = \mathrm{T}_1 \circ \bigvee_{i\in I}\langle a_i\rangle = \bigvee_{i\in I} \mathrm{T}_1 \circ \langle a_i\rangle = \bigvee_{i\in I}\langle \mathrm{T}_1(a_i)\rangle$. Thus, we have

$$0 = (\bigvee_{i\in I}\langle \mathrm{T}_1(a_i)\rangle)^{\perp} = \bigwedge_{i\in I}\langle (\mathrm{T}_1(a_i))^{\perp}\rangle.$$

Then, there is no homomorphism $v : A \to \mathbf{n}$ such that $v((\mathrm{T}_1(a_i))^{\perp}) = 1$ for any $i \in I$. Therefore, by Proposition 2.17, there is no prime $\mathbf{n}$-filter of A which contains $\{(\mathrm{T}_1(a_i))^{\perp}\ ;\ i \in I\}$. Thus, by Corollary 2.16, $\{(\mathrm{T}_1(a_i))^{\perp}\ ;\ i \in I\}$ does not have f.i.p. with respect to $*$ and so there is a finite subset $\{i_1, ... i_m\}$ of I such that $(\mathrm{T}_1(a_{i_1}))^{\perp} * ... * (\mathrm{T}_1(a_{i_m}))^{\perp} = 0$, whence $\mathrm{T}_1(a_{i_1})\wp ... \wp \mathrm{T}_1(a_{i_m}) = 1$. Since $\mathrm{T}_1(a_{i_k})$ is idempotent for any $k \in \{1, ..., m\}$, we have $\mathrm{T}_1(a_{i_1}) \vee ... \vee \mathrm{T}_1(a_{i_m}) = 1$ and, by Lemma 2.7, $\mathrm{T}_1(a_{i_1} \vee ... \vee a_{i_m}) = 1$. By $\mathrm{T}_1(x) \leq x$, we have $a_{i_1} \vee ... \vee a_{i_m} = 1$, whence $\langle a_{i_1} \vee ... \vee a_{i_m}\rangle = 1$. This completes the proof. ∎

Lemma 4.13. *Let A be an Ł_n^c-algebra. Then,* $\mathrm{Spec}(A)$ *is Kolmogorov.*

Proof. Let $v_1, v_2 \in \mathrm{Spec}(A)$ with $v_1 \neq v_2$. Then there is $a \in A$ such that $v_1(a) \neq v_2(a)$, whence we have $\langle a\rangle(v_1) \neq \langle a\rangle(v_2)$. □

Lemma 4.14. *Let A be an Ł_n^c-algebra. Then,* $\mathrm{Spec}(A)$ *is zero-dimensional.*

Proof. Since $\{\langle a\rangle\ ;\ a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$, it suffices to show that

$$\mathrm{Cont} \circ \mathrm{Spec}(A) = \{\langle a\rangle\ ;\ a \in A\}.$$

We first show that $\mathrm{Cont} \circ \mathrm{Spec}(A) \supset \{\langle a\rangle\ ;\ a \in A\}$, i.e., $\langle a\rangle$ is continuous for any $a \in A$. Let $a \in A$ and μ an $\mathbf{n}$-fuzzy set on $\mathbf{n}$. Then, by Lemma 2.9,

$$\langle a\rangle^{-1}(\mu) = \mu \circ \langle a\rangle = \bigvee_{r\in\mathbf{n}} (\mathrm{S}_{\mu(r)} \circ \mathrm{T}_r) \circ \langle a\rangle = \langle \bigvee_{r\in\mathbf{n}} (\mathrm{S}_{\mu(r)}(\mathrm{T}_r(a)))\rangle.$$

Hence $\langle a\rangle$ is continuous.

Next we show $\mathrm{Cont} \circ \mathrm{Spec}(A) \subset \{\langle a\rangle\ ;\ a \in A\}$. Let $f \in \mathrm{Cont} \circ \mathrm{Spec}(A)$ and $r \in \mathbf{n}$. Define an $\mathbf{n}$-fuzzy set λ_r on $\mathbf{n}$ by $\lambda_r(x) = 1$ for $x = r$ and $\lambda_r(x) = 0$ for $x \neq r$. Since f is continuous, $f^{-1}(\lambda_r) = \bigvee_{i\in I}\langle a_i\rangle$ for some $a_i \in A$. Now the following holds:

$$1 = f^{-1}(\lambda_r) \vee (f^{-1}(\lambda_r))^{\perp} = (\bigvee_{i\in I}\langle a_i\rangle) \vee (f^{-1}(\lambda_r))^{\perp}.$$

Here, we have $(f^{-1}(\lambda_r))^{\perp} = (\lambda_r \circ f)^{\perp} = {\lambda_r}^{\perp} \circ f = f^{-1}({\lambda_r}^{\perp})$. Since $f^{-1}({\lambda_r}^{\perp})$ is an open $\mathbf{n}$-fuzzy set, $(f^{-1}(\lambda_r))^{\perp}$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}(A)$. Since $\mathrm{Spec}(A)$ is compact by Lemma 4.12, there is a finite subset J of I such that $1 = (\bigvee_{j\in J}\langle a_j\rangle) \vee (f^{-1}(\lambda_r))^{\perp}$. Thus, $f^{-1}(\lambda_r) \leq \bigvee_{j\in J}\langle a_j\rangle$. Since $\bigvee_{j\in J}\langle a_j\rangle \leq \bigvee_{i\in I}\langle a_i\rangle = f^{-1}(\lambda_r)$, we have $f^{-1}(\lambda_r) = \bigvee_{j\in J}\langle a_j\rangle$. Since J is finite, $f^{-1}(\lambda_r) = \bigvee_{j\in J}\langle a_j\rangle = \langle\bigvee_{j\in J} a_j\rangle$. Let $a_r = \bigvee_{j\in J} a_j$. Note that if $v \in f^{-1}(\{r\})$ then $v(a_r) = 1$ and that if $v \notin f^{-1}(\{r\})$ then $v(a_r) = 0$. We claim that $f = \langle\bigvee_{r\in\mathbf{n}}(r \wedge a_r)\rangle$. If $v \in f^{-1}(\{s\})$ for $s \in \mathbf{n}$, then

$$\langle\bigvee_{r\in\mathbf{n}}(r \wedge a_r)\rangle(v) = v(\bigvee_{r\in\mathbf{n}}(r \wedge a_r)) = \bigvee_{r\in\mathbf{n}}(r \wedge v(a_r)) = s = f(v).$$

This completes the proof. □

By the above lemmas, we obtain the following proposition.

Proposition 4.15. *Let A be an object in* Ł_n^c-Alg. *Then,* $\mathrm{Spec}(A)$ *is an object in the category* FBS_n.

Proposition 4.16. *Let* A_1 *and* A_2 *be objects in* $Ł_n^c$-Alg *and* $f : A_1 \to A_2$ *an arrow in* $Ł_n^c$-Alg*. Then,* $\mathrm{Spec}(f)$ *is an arrow in* FBS_n*.*

Proof. Since the inverse image $(\mathrm{Spec}(f))^{-1}$ commutes with $\bigvee$, it suffices to show that $(\mathrm{Spec}(f))^{-1}(\langle a\rangle)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}(A_2)$ for any $a \in A_1$. For $v \in \mathrm{Spec}(A_2)$, we have

$$(\mathrm{Spec}(f)^{-1}(\langle a\rangle))(v) = \langle a\rangle \circ \mathrm{Spec}(f)(v) = \langle a\rangle(v \circ f) = v \circ f(a) = \langle f(a)\rangle(v).$$

Hence $(\mathrm{Spec}(f))^{-1}(\langle a\rangle) = \langle f(a)\rangle$, which is an open $\mathbf{n}$-fuzzy set. □

Definition 4.17. We define a contravariant functor $\mathrm{Cont} : \mathrm{FBS}_n \to Ł_n^c$-Alg.

For an object S in FBS_n, $\mathrm{Cont}(S)$ is defined as in Definition 4.2.

For an arrow $f : S \to T$ in FBS_n, $\mathrm{Cont}(f) : \mathrm{Cont}(T) \to \mathrm{Cont}(S)$ is defined by $\mathrm{Cont}(f)(g) = g \circ f$ for $g \in \mathrm{Cont}(T)$.

Since the operations of $\mathrm{Cont}(S)$ are defined pointwise, $\mathrm{Cont}(S)$ is an $Ł_n^c$-algebra and the following holds, whence Cont is well-defined.

Proposition 4.18. *Let* S_1 *and* S_2 *be objects in* FBS_n*, and* $f : S_1 \to S_2$ *an arrow in* FBS_n*. Then,* $\mathrm{Cont}(f)$ *is an arrow in* $Ł_n^c$-Alg*.*

Definition 4.19. Let A be an $Ł_n^c$-algebra. Then, $\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ is defined as the set of all homomorphisms of Boolean algebras from $\mathcal{B}(A)$ to $\mathbf{2}$ equipped with the (ordinary) topology generated by $\{\langle a\rangle_{\mathbf{2}}\ ;\ a \in \mathcal{B}(A)\}$, where $\langle a\rangle_{\mathbf{2}} = \{v \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))\ ;\ v(a) = 1\}$.

Proposition 4.20. *Let* A *be an* $Ł_n^c$*-algebra. Define a function* t_1 *from* $\mathrm{Spec}(A)^*$ *to* $\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$ *by* $t_1(v) = \mathrm{T}_1 \circ v$*. Then,* t_1 *is a homeomorphism.*

Proof. By Lemma 2.10, t_1 is injective. We show that t_1 is surjective. Let $v \in \mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A))$. Define $u \in \mathrm{Spec}(A)$ by $u(a) = r \Leftrightarrow \mathrm{T}_r(a) \in v^{-1}(\{1\})$ for $a \in A$, where note $\mathrm{T}_r(a) \in \mathcal{B}(A)$. Then, in a similar way to Proposition 2.17, it is verified that u is a homomorphism (i.e., $u \in \mathrm{Spec}(A)$). Moreover, we have $t_1(u) = v$ on $\mathcal{B}(A)$. Thus t_1 is bijective. It is straightforward to verify the remaining part of the proof. Note that, for $\langle a\rangle_{\mathbf{n}} = \{v \in \mathrm{Spec}(A)\ ;\ v(a) = 1\}$, $\{\langle a\rangle_{\mathbf{n}}\ ;\ a \in A\}$ forms an open basis of $\mathrm{Spec}(A)^*$ and that $t_1(\langle a\rangle_{\mathbf{n}}) = \langle \mathrm{T}_1(a)\rangle_{\mathbf{2}}$ for $a \in A$. □

4.3 A fuzzy topological duality for $Ł_n^c$-algebras

Theorem 4.21. *Let* A *be an* $Ł_n^c$*-algebra. Then, there is an isomorphism between* A *and* $\mathrm{Cont} \circ \mathrm{Spec}(A)$ *in the category* $Ł_n^c$-Alg*.*

Proof. Define $\langle\text{-}\rangle : A \to \mathrm{Cont} \circ \mathrm{Spec}(A)$ as in Definition 4.9. In the proof of Lemma 4.14, it has already been proven that $\langle\text{-}\rangle$ is well-defined and surjective. Since the operations of $\mathrm{Cont} \circ \mathrm{Spec}(A)$ are defined pointwise, $\langle\text{-}\rangle$ is a homomorphism.

Thus it suffices to show that $\langle\text{-}\rangle$ is injective. Assume that $\langle a\rangle = \langle b\rangle$ for $a, b \in A$, which means that, for any $v \in \mathrm{Spec}(A)$, we have $v(a) = v(b)$. Thus, for any $v \in \mathrm{Spec}(A)$ and any $r \in \mathbf{n}$, we have $v(\mathrm{T}_r(a)) = v(\mathrm{T}_r(b))$. Thus, it follows from Proposition 2.17 that, for any prime $\mathbf{n}$-filter P of A and any $r \in \mathbf{n}$, $\mathrm{T}_r(a) \in P$ iff $\mathrm{T}_r(b) \in P$.

We claim that $\mathrm{T}_r(a) = \mathrm{T}_r(b)$ for any $r \in \mathbf{n}$. Suppose for contradiction that $\mathrm{T}_r(a) \neq \mathrm{T}_r(b)$ for some $r \in \mathbf{n}$. We may assume without loss of generality that $\mathrm{T}_r(a) \not\leq \mathrm{T}_r(b)$. Let $F = \{x \in A\ ;\ \mathrm{T}_r(a) \leq x\}$. Then, since $\mathrm{T}_r(a)$ is idempotent, F is an $\mathbf{n}$-filter of A. Cleary, $\mathrm{T}_r(b) \notin F$. Thus, by Lemma 2.14, there is a prime $\mathbf{n}$-filter P of A such that $F \subset P$ and $\mathrm{T}_r(b) \notin P$. By $F \subset P$, we have $\mathrm{T}_r(a) \in P$, which contradicts $\mathrm{T}_r(b) \notin P$, since we have already shown that $\mathrm{T}_r(a) \in P$ iff $\mathrm{T}_r(b) \in P$. Thus, $\mathrm{T}_r(a) = \mathrm{T}_r(b)$ for any $r \in \mathbf{n}$, whence $\bigwedge_{r\in\mathbf{n}}(\mathrm{T}_r(a) \leftrightarrow \mathrm{T}_r(b)) = 1$. Hence, it follows from Lemma 2.11 that $a = b$, and therefore $\langle\text{-}\rangle$ is injective. □

Theorem 4.22. *Let S be an $\mathbf{n}$-fuzzy Boolean space. Then, there is an isomorphism between S and $\mathrm{Spec} \circ \mathrm{Cont}(S)$ in the category FBS_n.*

Proof. Define $\Psi : S \to \mathrm{Spec} \circ \mathrm{Cont}(S)$ by $\Psi(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{Cont}(S)$. Since the operations of $\mathrm{Cont}(S)$ are defined pointwise, $\Psi(x)$ is a homomorphism and so Ψ is well-defined.

We show that Ψ is continuous. Let $f \in \mathrm{Cont}(S)$. Then $\Psi^{-1}(\langle f\rangle) = f$ by the following:

$$(\Psi^{-1}(\langle f\rangle))(x) = \langle f\rangle \circ \Psi(x) = \Psi(x)(f) = f(x).$$

Since $f \in \mathrm{Cont}(S)$ and S is zero-dimensional, f is an an open $\mathbf{n}$-fuzzy set and so $\Psi^{-1}(\langle f\rangle)$ is an open $\mathbf{n}$-fuzzy set on S. Since the inverse image Ψ^{-1} commutes with $\bigvee$, it follows that Ψ is continuous.

Next we show that Ψ is injective. Let $x, y \in S$ with $x \neq y$. Since S is Kolmogorov and zero-dimensional, there is $f \in \mathrm{Cont}(S)$ with $f(x) \neq f(y)$. Thus, $\Psi(x)(f) = f(x) \neq f(y) = \Psi(y)(f)$, whence Ψ is injective.

Next we show that Ψ is surjective. Let $v \in \mathrm{Spec} \circ \mathrm{Cont}(S)$. Consider $\{f^{-1}(\{1\})\ ;\ v(f) = 1\}$. Define $\mu : \mathbf{n} \to \mathbf{n}$ by $\mu(1) = 0$ and $\mu(x) = 1$ for $x \neq 1$. Since $f^{-1}(\mu)$ $(= \mu \circ f)$ is an open $\mathbf{n}$-fuzzy set on S for $f \in \mathrm{Cont}(S)$, $(\mu \circ f)^{-1}(\{1\})$ is an open subset of S^*. Since $(\mu \circ f)^{-1}(\{1\}) = (f^{-1}(\{1\}))^c$, $f^{-1}(\{1\})$ is a closed subset of S^* for $f \in \mathrm{Cont}(S)$.

We claim that $\{f^{-1}(\{1\})\,;\,v(f)=1\}$ has the finite intersection property. Since $f^{-1}(\{1\})\cap g^{-1}(\{1\})=(f\wedge g)^{-1}(\{1\})$ for $f,g\in \mathrm{Cont}(S)$, it suffices to show that if $v(f)=1$ then $f^{-1}(\{1\})$ is not empty. Suppose for contradiction that $v(f)=1$ and $f^{-1}(\{1\})=\emptyset$. Since $f^{-1}(\{1\})=\emptyset$, we have $\mathrm{T}_1(f)=0$. Thus $v(\mathrm{T}_1(f))=0$ and so $v(f)\neq 1$, which contradicts $v(f)=1$.

By Proposition 4.8, S^* is compact. Thus, there is $z\in S$ such that $z\in\bigcap\{f^{-1}(\{1\})\,;\,v(f)=1\}$. We claim that $\Psi(z)=v$. By the definition of z, if $v(f)=1$ then $\Psi(z)(f)=1$. We show the converse. Suppose for constradiction that $\Psi(z)(f)=1$ and $v(f)\neq 1$. Then $v(\mathrm{T}_1(f))=\mathrm{T}_1(v(f))=0$ and so $v((\mathrm{T}_1(f))^{\perp})=1$. By the definition of z, $(\mathrm{T}_1(f))^{\perp}(z)=1$ and so $(\mathrm{T}_1(f))(z)=0$. Thus $f(z)\neq 1$, which contradicts $\Psi(z)(f)=1$. Hence, for any $f\in\mathrm{Cont}(S)$, $v(f)=1$ iff $\Psi(z)(f)=1$. By Lemma 2.10, we have $\Psi(z)=v$. Hence, Ψ is surjective.

Finally we show that Ψ^{-1} is an arrow in the category FBS_n. It suffices to show that, for any open $\mathbf{n}$-fuzzy set λ on S, $\Psi(\lambda)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}\circ\mathrm{Cont}(S)$. Since S is zero-dimensional, there are $f_i\in\mathrm{Cont}(S)$ with $\lambda=\bigvee_{i\in I}f_i$. For $v\in\mathrm{Spec}\circ\mathrm{Cont}(S)$, the following holds:

$$\Psi(\lambda)(v)=\bigvee\{\lambda(x);x\in\Psi^{-1}(\{v\})\}=\lambda(z)=v(\lambda)=v(\bigvee_{i\in I}f_i)=(\bigvee_{i\in I}\langle f_i\rangle)(v),$$

where z is defined as the unique element x such that $\Psi(x)=v$ (for the definition of the direct image of an $\mathbf{n}$-fuzzy set, see Subsection 3.1). Hence $\Psi(\lambda)=\bigvee_{i\in I}\langle f_i\rangle$ and so $\Psi(\lambda)$ is an open $\mathbf{n}$-fuzzy set on $\mathrm{Spec}\circ\mathrm{Cont}(S)$. □

By Theorem 4.21 and Theorem 4.22, we obtain a fuzzy topological duality for Ł^c_n-algebras, which is a generalization of Stone duality for Boolean algebras to the n-valued case via fuzzy topology.

Theorem 4.23. *The category* Ł^c_n-Alg *is dually equivalent to the category* FBS_n *via the functors* Spec *and* Cont*.*

Proof. Let Id_1 denote the identity functor on Ł^c_n-Alg and Id_2 denote the identity functor on FBS_n. Then, we define two natural transformations $\epsilon:\mathrm{Id}_1\to\mathrm{Cont}\circ\mathrm{Spec}$ and $\eta:\mathrm{Id}_2\to\mathrm{Spec}\circ\mathrm{Cont}$. For an Ł^c_n-algebra A, define $\epsilon_A:A\to\mathrm{Cont}\circ\mathrm{Spec}(A)$ by $\epsilon_A=\langle\text{-}\rangle$ (see Theorem 4.21). For an $\mathbf{n}$-fuzzy Boolean space S, define $\eta_S:S\to\mathrm{Spec}\circ\mathrm{Cont}(S)$ by $\eta_S=\Psi$ (see Theorem 4.22). It is straightforward to see that η and ϵ are natural transformations. By Theorem 4.21 and Theorem 4.22, η and ϵ are natural isomorphisms. □

5 MŁ_n^c-algebras and basic properties

We define modal Łukasiewicz n-valued logic with truth constants MŁ_n^c by $\mathbf{n}$-valued Kripke semantics. The connectives of MŁ_n^c are a unary connective $\Box$ and the connectives of Ł_n^c. $\mathbf{Form}_\Box$ denotes the set of formulas of MŁ_n^c.

Definition 5.1. Let (W, R) be a Kripke frame (i.e., R is a relation on a set W). Then, e is a Kripke $\mathbf{n}$-valuation on (W, R) iff e is a function from $W \times \mathbf{Form}_\Box$ to $\mathbf{n}$ which satisfies: For each $w \in W$ and $\varphi, \psi \in \mathbf{Form}_\Box$,

- $e(w, \Box\varphi) = \bigwedge\{e(w', \varphi) \,;\, wRw'\}$;
- $e(w, \varphi@\psi) = e(w, \varphi)@e(w, \psi)$ for $@ = \wedge, \vee, *, \wp, \to$;
- $e(w, \varphi^\perp) = (e(w, \varphi))^\perp$;
- $e(w, r) = r$ for $r \in \mathbf{n}$.

Then, (W, R, e) is called an $\mathbf{n}$-valued Kripke model. Define MŁ_n^c as the set of all those formulas $\varphi \in \mathbf{Form}_\Box$ such that $e(w, \varphi) = 1$ for any $\mathbf{n}$-valued Kripke model (W, R, e) and any $w \in W$.

By straightforward computation, we have the following lemma. Recall the definition of U_r (Definition 2.8).

Lemma 5.2. *Let $\varphi, \psi \in \mathbf{Form}_\Box$ and $r \in \mathbf{n}$. (i) $\mathrm{U}_r(\Box\varphi) \leftrightarrow \Box\mathrm{U}_r(\varphi) \in \text{MŁ}_n^c$. (ii) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \in \text{MŁ}_n^c$ and $\Box 1 \leftrightarrow 1 \in \text{MŁ}_n^c$. (iii) $\Box(\varphi * \varphi) \leftrightarrow (\Box\varphi) * (\Box\varphi) \in \text{MŁ}_n^c$ and $\Box(\varphi \wp \varphi) \leftrightarrow (\Box\varphi)\wp(\Box\varphi) \in \text{MŁ}_n^c$.*

Definition 5.3. For $X \subset \mathbf{Form}_\Box$, X is satisfiable iff there are an $\mathbf{n}$-valued Kripke model (W, R, e) and $w \in W$ such that $e(w, \varphi) = 1$ for any $\varphi \in X$.

MŁ_n^c-algebras and homomorphisms are defined as follows.

Definition 5.4. Let A be an Ł_n^c-algebra. Then, $(A, \Box)$ is an MŁ_n^c-algebra iff it satisfies the following set of equations: $\{\varphi = \psi \,;\, \varphi \leftrightarrow \psi \in \text{MŁ}_n^c\}$.

A homomorphism of MŁ_n^c-algebras is defined as a homomorphism of Ł_n^c-algebras which additionally preserves the operation $\Box$.

Throughout this paper, we do not distinguish between formulas of MŁ_n^c and terms of MŁ_n^c-algebras.

Definition 5.5. Let A be an MŁ_n^c-algebra. Define a relation $R_\Box$ on $\mathrm{Spec}(A)$ by

$$vR_\Box u \Leftrightarrow \forall r \in \mathbf{n}\, \forall x \in A\ (v(\Box x) \geq r \text{ implies } u(x) \geq r).$$

Define $e : \mathrm{Spec}(A) \times A \to \mathbf{n}$ by $e(v, x) = v(x)$ for $v \in \mathrm{Spec}(A)$ and $x \in A$. Then, $(\mathrm{Spec}(A), R_\Box, e)$ is called the $\mathbf{n}$-valued canonical model of A.

Proposition 5.6. *Let A be an* $\mathrm{MŁ}_n^c$*-algebra. Then, the* $\mathbf{n}$*-valued canonical model* $(\mathrm{Spec}(A), R_\Box, e)$ *of A is an* $\mathbf{n}$*-valued Kripke model. In particular,* $e(v, \Box x) = v(\Box x) = \bigwedge\{u(x)\,;\, vR_\Box u\}$ *for $x \in A$ and $v \in \mathrm{Spec}(A)$.*

Proof. It suffices to show that e is a Kripke $\mathbf{n}$-valuation. Since v is a homomorphism of $\mathrm{Ł}_n^c$-algebras, it remains to show $e(v, \Box x) = \bigwedge\{u(x)\,;\, vR_\Box u\}$. To prove this, it is enough to show that, for any $r \in \mathbf{n}$, (i) $v(\Box x) \geq r$ iff (ii) $vR_\Box u$ implies $u(x) \geq r$. By the definition of $R_\Box$, (i) implies (ii). We show the converse. To prove the contrapositive, assume $v(\Box x) \not\geq r$, i.e., $\mathrm{U}_r(\Box x) \notin v^{-1}(\{1\})$. Let

$$F_0 = \{\mathrm{U}_s(x)\,;\, s \in \mathbf{n} \text{ and } \mathrm{U}_s(\Box x) \in v^{-1}(\{1\})\}.$$

Let F be the $\mathbf{n}$-filter of A generated by F_0. We claim that $\mathrm{U}_r(x) \notin F$. Suppose for contradiction that $\mathrm{U}_r(x) \in F$. Then, there is $\varphi \in A$ such that $\varphi \leq \mathrm{U}_r(x)$ and φ is constructed from $*$ and elements of F_0. Since $\mathrm{U}_s(x)$ is idempotent, $\mathrm{U}_{s_1}(x_1) * \mathrm{U}_{s_2}(x_2) = \mathrm{U}_{s_1}(x_1) \wedge \mathrm{U}_{s_2}(x_2)$ and so we may assume that $\varphi = \bigwedge\{\mathrm{U}_s(x)\,;\, \mathrm{U}_s(x) \in F_1\}$ for some finite subset F_1 of F_0. By Lemma 5.2, $\Box\varphi = \bigwedge\{\mathrm{U}_s(\Box x)\,;\, \mathrm{U}_s(x) \in F_1\}$. By the definition of F_0, $\mathrm{U}_s(\Box x) \in v^{-1}(\{1\})$ for any $\mathrm{U}_s(x) \in F_1$ and so $\Box\varphi \in v^{-1}(\{1\})$. Since $\varphi \leq \mathrm{U}_r(x)$, we have $\Box\varphi \leq \Box\mathrm{U}_r(x) = \mathrm{U}_r(\Box x)$. Thus, $\mathrm{U}_r(\Box x) \in v^{-1}(\{1\})$, which contradicts $\mathrm{U}_r(\Box x) \notin v^{-1}(\{1\})$. Hence $\mathrm{U}_r(x) \notin F$. By Proposition 2.14, there is a prime $\mathbf{n}$-filter P of A such that $\mathrm{U}_r(x) \notin P$ and $F \subset P$. By Proposition 2.17, $v_P \in \mathrm{Spec}(A)$. Since $\mathrm{U}_r(x) \notin P$, we have $v_P(x) \not\geq r$. Since $F_0 \subset F \subset P$, we have $vR_\Box v_P$. Thus, (ii) does not hold. □

The following is a compactness theorem for $\mathrm{MŁ}_n^c$.

Theorem 5.7. *Let* $X \subset \mathbf{Form}_\Box$*. Assume that any finite subset of X is satisfiable. Then, X is satisfiable.*

Proof. Let A be the Lindenbaum algebra of $\mathrm{MŁ}_n^c$. We may consider $X \subset A$. We show that X has f.i.p. with respect to $*$. If not, then there are $n \in \omega$ with $n \neq 0$ and $x_1, ..., x_n \in X$ such that $x_1 * ... * x_n = 0$, which is a contradiction, since $\{x_1, ..., x_n\}$ is satisfiable by assumption. Thus, by Proposition 2.16, there is a prime $\mathbf{n}$-filter P of A with $X \subset P$. By Proposition 2.17, v_P is a homomorphism, i.e., $v_P \in \mathrm{Spec}(A)$. Consider the $\mathbf{n}$-valued canonical model $(\mathrm{Spec}(A), R_\Box, e)$ of A. Then, $e(v_P, x) = v_P(x) = 1$ for any $x \in X$ by Proposition 2.17. Thus, X is satisfiable. □

Proposition 5.8. *Let A be an* $\mathrm{MŁ}_n^c$*-algebra. Then, $\mathcal{B}(A)$ forms a modal algebra.*

Proof. If $x \in A$ is idempotent, then $\Box x$ is also idempotent, since $\Box x * \Box x = \Box(x * x) = \Box x$ by Lemma 5.2. Thus, $\mathcal{B}(A)$ is closed under $\Box$. By Lemma 5.2, $\mathcal{B}(A)$ forms a modal algebra. $\Box$

Definition 5.9. Let A be an $\mathrm{MŁ}_n^c$-algebra. Define a relation $R_{\Box_\mathbf{2}}$ on $\mathrm{Spec}_\mathbf{2}(\mathcal{B}(A))$ by $vR_{\Box_\mathbf{2}}u \Leftrightarrow \forall x \in \mathcal{B}(A)\ (v(\Box x) = 1$ implies $u(x) = 1)$.

Proposition 5.10. *Let A be an $\mathrm{MŁ}_n^c$-algebra. For $v, u \in \mathrm{Spec}(A)$, $vR_\Box u$ iff $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$ (for the definition of t_1, see Proposition 4.20).*

Proof. By $\Box \mathrm{T}_1(x) = \mathrm{T}_1(\Box x)$, if $vR_\Box u$ then $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$. We show the converse. Assume $t_1(v)R_{\Box_\mathbf{2}}t_1(u)$. In order to show $vR_\Box u$, it suffices to prove that, for any $r \in \mathbf{n}$ and any $x \in A$, $v(\Box \mathrm{U}_r(x)) = 1$ implies $u(\mathrm{U}_r(x)) = 1$, which follows from the assumption, since we have $\mathrm{U}_r(x) \in \mathcal{B}(A)$ and $\mathrm{T}_1(\mathrm{U}_r(x)) = \mathrm{U}_r(x)$. $\Box$

6 A fuzzy topological duality for $\mathrm{MŁ}_n^c$-algebras

In this section, based on the fuzzy topological duality for $\mathrm{Ł}_n^c$-algebras, we show a fuzzy topological duality for $\mathrm{MŁ}_n^c$-algebras, which is a generalization of Jónsson-Tarski duality for modal algebras via fuzzy topology, where note that $\mathrm{MŁ}_2^c$-algebras coincide with modal algebras.

Definition 6.1. $\mathrm{MŁ}_n^c$-Alg denotes the category of $\mathrm{MŁ}_n^c$-algebras and homomorphisms of $\mathrm{MŁ}_n^c$-algebras.

Our aim in this section is to show that the category $\mathrm{MŁ}_n^c$-Alg is dually equivalent to the category FRS_n, which is defined in Definition 6.3 below.

For a Kripke frame (S, R), we can define a modal operator $\Box$ on the "$\mathbf{n}$-valued powerset algebra" $\mathbf{n}^S$ of S as follows.

Definition 6.2. Let (S, R) be a Kripke frame and f a function from S to $\mathbf{n}$. Define $\Box_R f : S \to \mathbf{n}$ by $(\Box_R f)(x) = \bigwedge\{f(y)\ ;\ xRy\}$.

Recall: For a Kripke frame (S, R) and an $\mathbf{n}$-fuzzy set μ on S, an $\mathbf{n}$-fuzzy set $R^{-1}[\mu]$ on S is defined by $R^{-1}[\mu](x) = \bigvee\{\mu(y)\ ;\ xRy\}$ for $x \in S$.

Definition 6.3. We define the category FRS_n as follows.

An object in FRS_n is a tuple (S, R) such that S is an object in FBS_n and that a relation R on S satisfies the following conditions:

1. if $\forall f \in \mathrm{Cont}(S)((\Box_R f)(x) = 1 \Rightarrow f(y) = 1)$ then xRy;

2. if $\mu \in \mathrm{Cont}(S)$, then $R^{-1}[\mu] \in \mathrm{Cont}(S)$.

An arrow $f : (S_1, R_1) \to (S_2, R_2)$ in FRS_n is an arrow $f : S_1 \to S_2$ in FBS_n which satisfies the following conditions:

1. if xR_1y then $f(x)R_2f(y)$;

2. if $f(x_1)R_2x_2$ then there is $y_1 \in S_1$ such that $x_1R_1y_1$ and $f(y_1) = x_2$.

An object in FRS_n is called an **n**-fuzzy relational space.

The item 1 in the object part of Definition 6.3 is an **n**-fuzzy version of the tightness condition of descriptive general frames in classical modal logic (for the definition of the tightness condition in classical modal logic, see [3]).

Definition 6.4. We define a contravariant functor $\mathrm{RSpec} : \mathrm{MŁ}^c_n\text{-Alg} \to \mathrm{FRS}_n$. For an object A in $\mathrm{MŁ}^c_n$-Alg, define $\mathrm{RSpec}(A) = (\mathrm{Spec}(A), R_\Box)$. For an arrow $f : A \to B$ in $\mathrm{MŁ}^c_n$-Alg, define $\mathrm{RSpec}(f) : \mathrm{RSpec}(B) \to \mathrm{RSpec}(A)$ by $\mathrm{RSpec}(f)(v) = v \circ f$ for $v \in \mathrm{Spec}(B)$.

We call $\mathrm{RSpec}(A)$ the relational spectrum of A. The well-definedness of RSpec is shown by Proposition 6.6 and Proposition 6.7 below.

Definition 6.5. Let A be an $\mathrm{MŁ}^c_n$-algebra. Then, we define $\mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A))$ as $(\mathrm{Spec}_{\mathbf{2}}(\mathcal{B}(A)), R_{\Box_{\mathbf{2}}})$. Let A_1 and A_2 be $\mathrm{MŁ}^c_n$-algebras and $f : \mathcal{B}(A_1) \to \mathcal{B}(A_2)$. Then, we define $\mathrm{RSpec}_{\mathbf{2}}(f) : \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2)) \to \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_1))$ by $\mathrm{RSpec}_{\mathbf{2}}(f)(v) = v \circ f$ for $v \in \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2))$.

Proposition 6.6. *For an $\mathrm{MŁ}^c_n$-algebra A, $\mathrm{RSpec}(A)$ is an object in FRS_n.*

Proof. It suffices to show the items 1 and 2 in the object part of Definition 6.3. We first show the item 1 by proving the contrapositive. Assume $(v, u) \notin R_\Box$, i.e., there are $r \in \mathbf{n}$ and $x \in A$ such that $v(\Box x) \geq r$ and $u(x) \ngeq r$. By Lemma 2.8, $v(\mathrm{U}_r(\Box x)) = 1$ and $u(\mathrm{U}_r(x)) = 0$. Then, $\langle \mathrm{U}_r(x)\rangle(u) = 0$. By Proposition 5.6 and Lemma 5.2,

$$(\Box_R\langle \mathrm{U}_r(x)\rangle)(v) = \bigwedge\{\langle \mathrm{U}_r(x)\rangle(v')\ ;\ vR_\Box v'\} = v(\Box \mathrm{U}_r(x)) = v(\mathrm{U}_r\Box x) = 1.$$

As is shown in the proof of Lemma 4.14, $\langle \mathrm{U}_r(x)\rangle$ is continuous.

We show the item 2. Since $\mathrm{Cont} \circ \mathrm{Spec}(A) = \{\langle x\rangle\ ;\ x \in A\}$ as is shown in the proof of Lemma 4.14, it suffices to show that, for any $x \in A$, $R_\Box^{-1}(\langle x\rangle) \in \mathrm{Cont}\circ\mathrm{Spec}(A)$. Let $\Diamond x$ denote $(\Box(x^\perp))^\perp$. Since $(R_\Box^{-1}(\langle x\rangle))(v) = \bigvee\{u(x)\ ;\ vR_\Box u\} = v(\Diamond x)$, we have $R_\Box^{-1}(\langle x\rangle) = \langle \Diamond x\rangle \in \mathrm{Cont} \circ \mathrm{Spec}(A)$. □

Proposition 6.7. *For* $\mathrm{MŁ}_n^c$*-algebras* A_1 *and* A_2*, let* $f : A_1 \to A_2$ *be a homomorphism of* $\mathrm{MŁ}_n^c$*-algebras. Then,* $\mathrm{RSpec}(f)$ *is an arrow in* FRS_n*.*

Proof. Define $f_* : \mathcal{B}(A_1) \to \mathcal{B}(A_2)$ by $f_*(x) = f(x)$ for $x \in \mathcal{B}(A_1)$. By Proposition 5.8, f_* is a homomorphism of modal algebras. Consider $\mathrm{RSpec}_{\mathbf{2}}(f_*) : \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2)) \to \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_1))$. By Jónsson-Tarski duality for modal algebras (see [16, 1]), $\mathrm{RSpec}_{\mathbf{2}}(f_*)$ is an arrow in FRS_2.

We first show that $\mathrm{RSpec}(f)$ satisfies the item 2 in the arrow part of Definition 6.3. Assume $\mathrm{RSpec}(f)(v_2) R_\Box u_1$ for $v_2 \in \mathrm{RSpec}(A_2)$ and $u_1 \in \mathrm{RSpec}(A_1)$. By Proposition 5.10, $t_1(\mathrm{RSpec}(f)(v_2)) R_{\Box_{\mathbf{2}}} t_1(u_1)$. It follows from $t_1(\mathrm{RSpec}(f)(v_2)) = \mathrm{T}_1 \circ v_2 \circ f = \mathrm{RSpec}_{\mathbf{2}}(f_*)(t_1(v_2))$ that we have $\mathrm{RSpec}_{\mathbf{2}}(f_*)(t_1(v_2)) R_{\Box_{\mathbf{2}}} t_1(u_1)$. Since $\mathrm{RSpec}_{\mathbf{2}}(f_*)$ is an arrow in FRS_2, there is $u_2 \in \mathrm{RSpec}_{\mathbf{2}}(\mathcal{B}(A_2))$ such that $t_1(v_2) R_{\Box_{\mathbf{2}}} u_2$ and $\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2) = t_1(u_1)$. Define $u_2' \in \mathrm{RSpec}(A_2)$ by $u_2'(x) = r \Leftrightarrow u_2(\mathrm{T}_r(x)) = 1$. It is verified in a similar way to Proposition 2.17 that u_2' is a homomorphism.

We claim that $v_2 R_\Box u_2'$ and $\mathrm{RSpec}(f)(u_2') = u_1$. Let $x \in A_2$ and $r \in \mathbf{n}$. If $v_2(\Box x) \geq r$ then $(t_1(v_2))(\Box \mathrm{U}_r(x)) = 1$ and, since $t_1(v_2) R_{\Box_{\mathbf{2}}} u_2$, we have $u_2(\mathrm{U}_r(x)) = 1$, whence $u_2'(x) \geq r$. Thus, $v_2 R_\Box u_2'$. Next we show $\mathrm{RSpec}(f)(u_2') = u_1$. Let $r = (\mathrm{RSpec}(f)(u_2'))(x)$ for $x \in A_1$. Then, $u_2(\mathrm{T}_r(f(x))) = 1$ and so $(\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2))(\mathrm{T}_r(x)) = 1$. It follows from $\mathrm{RSpec}_{\mathbf{2}}(f_*)(u_2) = t_1(u_1)$ that $(t_1(u_1))(\mathrm{T}_r(x)) = 1$ and so $u_1(\mathrm{T}_r(x)) = 1$, whence $u_1(x) = r = (\mathrm{RSpec}(f)(u_2'))(x)$. Thus $\mathrm{RSpec}(f)$ satisfies the item 2.

It is easier to verify that $\mathrm{RSpec}(f)$ satisfies the item 1 in the arrow part of Definition 6.3. □

Definition 6.8. A contravariant functor $\mathrm{MCont} : \mathrm{FRS}_n \to \mathrm{MŁ}_n^c\text{-Alg}$ is defined as follows. For an object (S, R) in FRS_n, define $\mathrm{MCont}(S, R) = (\mathrm{Cont}(S), \Box_R)$. For an arrow $f : (S_1, R_1) \to (S_2, R_2)$ in FRS_n, define $\mathrm{MCont}(f) : \mathrm{MCont}(S_2, R_2) \to \mathrm{MCont}(S_1, R_1)$ by $\mathrm{MCont}(f)(g) = g \circ f$ for $g \in \mathrm{Cont}(S_2)$.

The well-definedness of MCont is shown by the following propositions.

Proposition 6.9. *For an object* (S, R) *in* FRS_n*,* $\mathrm{MCont}(S, R)$ *is an* $\mathrm{MŁ}_n^c$*-algebra.*

Proof. We first show that if $f \in \mathrm{Cont}(S)$ then $\Box_R f \in \mathrm{Cont}(S)$. Let $f \in \mathrm{Cont}(S)$ and μ an open $\mathbf{n}$-fuzzy set on $\mathbf{n}$. Define μ_r as in the proof of Lemma 4.3 and then it suffices to show that $(\Box_R f)^{-1}(\mu_r)$ is an open $\mathbf{n}$-fuzzy set on S for any $r \in \mathbf{n}$. By Lemma 2.8,

$$(\Box_R f)^{-1}(\mu_r) = R^{-1}[\mu_r \circ f] \wedge (R^{-1}[(\mathrm{U}_r \circ f)^\perp])^\perp.$$

Since both $\mu_r \circ f$ and $(\mathrm{U}_r \circ f)^{\perp}$ are elements of $\mathrm{Cont}(S)$, the right-hand side is an element of $\mathrm{Cont}(S)$ by the definition of R and so is an open $\mathbf{n}$-fuzzy set on S, since S is zero-dimensional. Thus $\Box_R f \in \mathrm{Cont}(S)$.

Next we show that $\mathrm{MCont}(S, R)$ satisfies $\{\varphi = \psi \,;\, \varphi \leftrightarrow \psi \in \mathrm{MŁ}_n^c\}$. Consider $\mathrm{Cont}(S)$ as the set of propositional variables. Since $\mathrm{Cont}(S)$ is closed under the operations of $\mathrm{Cont}(S)$, an element of $\mathbf{Form}_\Box$ may be seen as an element of $\mathrm{Cont}(S)$. Define $e : S \times \mathbf{Form}_\Box \to \mathbf{n}$ by $e(w, f) = f(w)$ for $w \in S$ and $f \in \mathrm{Cont}(S)$. Then, (S, R, e) is an $\mathbf{n}$-valued Kripke model by the definition of the operations of $\mathrm{Cont}(S)$. Since $e(w, f) = 1$ for any $w \in S$ iff $f = 1$, it follows from the definition of $\mathrm{MŁ}_n^c$ that $\mathrm{MCont}(S, R)$ satisfies $\{\varphi = \psi \,;\, \varphi \leftrightarrow \psi \in \mathrm{MŁ}_n^c\}$. $\Box$

Proposition 6.10. *Let $f : (S_1, R_1) \to (S_2, R_2)$ be an arrow in FRS_n. Then, $\mathrm{MCont}(f)$ is a homomorphism of $\mathrm{MŁ}_n^c$-algebras.*

Proof. It remains to show that $\mathrm{MCont}(f)(\Box g_2) = \Box(\mathrm{MCont}(f)(g_2))$ for $g_2 \in \mathrm{Cont}(S_2)$. For $x_1 \in S_1$, $(\mathrm{MCont}(f)(\Box g_2))(x_1) = \bigwedge\{g_2(y_2) \,;\, f(x_1) R_2 y_2\}$. Let a denote the right-hand side. We also have $(\Box(\mathrm{MCont}(f)(g_2)))(x_1) = \bigwedge\{g_2(f(y_1)) \,;\, x_1 R_1 y_1\}$. Let b denote the right-hand side. Since $x_1 R_1 y_1$ implies $f(x_1) R_1 f(y_1)$, we have $a \leq b$. By the item 2 in the arrow part of Definition 6.3, we have $a \geq b$. Hence $a = b$. $\Box$

Theorem 6.11. *Let A be an object in* $\mathrm{MŁ}_n^c$-Alg. *Then, A is isomorphic to* $\mathrm{MCont} \circ \mathrm{RSpec}(A)$ *in the category* $\mathrm{MŁ}_n^c$-Alg.

Proof. We claim that $\langle\text{-}\rangle : A \to \mathrm{MCont} \circ \mathrm{RSpec}(A)$ is an isomorphism of $\mathrm{MŁ}_n^c$-algebras. By Theorem 4.21, it remains to show that $\langle \Box x \rangle = \Box_{R_\Box} \langle x \rangle$ for $x \in A$. By Proposition 5.6, we have the following for $v \in \mathrm{Spec}(A)$: $(\Box_{R_\Box}\langle x \rangle)(v) = \bigwedge\{u(x) \,;\, v R_\Box u\} = v(\Box x) = \langle \Box x \rangle(v)$. $\Box$

Theorem 6.12. *Let (S, R) be an object in FRS_n. Then, (S, R) is isomorphic to $\mathrm{RSpec} \circ \mathrm{MCont}(S, R)$ in the category FRS_n.*

Proof. Define $\Phi : (S, R) \to \mathrm{RSpec} \circ \mathrm{MCont}(S, R)$ by $\Phi(x)(f) = f(x)$ for $x \in S$ and $f \in \mathrm{Cont}(S)$. We show: For any $x, y \in S$, xRy iff $\Phi(x) R_{\Box_R} \Phi(y)$. Assume xRy. Let $r \in \mathbf{n}$ and $f \in \mathrm{Cont}(S)$ with $\Phi(x)(\Box_R f) \geq r$. Since $\Phi(x)(\Box_R f) = \bigwedge\{f(z) \,;\, xRz\}$, we have $\Phi(y)(f) = f(y) \geq r$. Next we show the converse. To prove the contrapositive, assume $(x, y) \notin R$. By Definition 6.3, there is $f \in \mathrm{Cont}(S)$ such that $(\Box_R f)(x) = 1$ and $f(y) \neq 1$. Then, $\Phi(x)(\Box_R f) = 1$ and $\Phi(y)(f) \neq 1$. Thus, we have $(\Phi(x), \Phi(y)) \notin R_{\Box_R}$.

By Theorem 4.22, it remains to prove that Φ and Φ^{-1} satisfy the item 2 in the arrow part of Definition 6.3, which follows from the above fact that xRy iff $\Phi(x) R_{\Box_R} \Phi(y)$, since Φ is bijective. $\Box$

By Theorem 6.11 and Theorem 6.12, we obtain a fuzzy topological duality for $\mathrm{MŁ}_n^c$-algebras, which is a generalization of Jónsson-Tarski duality for modal algebras to the n-valued case via fuzzy topology.

Theorem 6.13. *The category* $\mathrm{MŁ}_n^c$-Alg *is dually equivalent to the category* FRS_n *via the functors* RSpec(-) *and* MCont(-).

Proof. By arguing as in the proof of Theorem 4.23, this theorem follows immediately from Theorem 6.11 and Theorem 6.12. □

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